

REPRESENTATIONS OF GL_N OVER FINITE LOCAL PRINCIPAL IDEAL RINGS - AN OVERVIEW

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ABSTRACT. We give a survey of the representation theory of GL_N over finite local principal ideal rings via Clifford theory, with an emphasis on the construction of regular representations. We review results of Shintani and Hill, and the generalisation of Takase. We then summarise the main features, with some details but without proofs, of the recent constructions of regular representations due to Krakovski–Onn–Singla and Stasinski–Stevens, respectively.

1. INTRODUCTION

This paper is a survey of the (complex) representation theory of the group $GL_N(\mathfrak{o})$, where \mathfrak{o} is a compact discrete valuation ring, or equivalently, the ring of integers in a non-Archimedean local field with finite residue field \mathbb{F}_q of characteristic p . Since $GL_N(\mathfrak{o})$ is a profinite group, we consider its continuous representations, and a representation is continuous if and only if it is smooth if and only if it factors through a finite quotient $GL_N(\mathfrak{o}_r)$, where $\mathfrak{o}_r := \mathfrak{o}/\mathfrak{p}^r$, \mathfrak{p} is the maximal ideal in \mathfrak{o} , and $r \geq 1$. We therefore focus on the representations of the finite groups $GL_N(\mathfrak{o}_r)$.

The representation theory of $GL_N(\mathfrak{o}_r)$ has a relatively long history (see the historical notes in Section 2), and has very recently seen intensified activity from several directions. We will focus mostly on the recent developments regarding so-called *regular* representations, studied via Clifford theory. Regular representations roughly correspond to regular conjugacy classes of matrices in the Lie algebra $\mathfrak{g}_r = M_N(\mathfrak{o}_r)$, that is, matrices whose centralisers mod \mathfrak{p} have dimension N . The first construction of this kind goes back to Shintani [28], who constructed all the regular representations when r is even. This was followed by work of Hill [12], who rediscovered Shintani’s construction and also provided a partial construction of so-called split regular representations for r odd. As we will see in subsequent sections, the representation theory of $GL_N(\mathfrak{o}_r)$ is much harder when r is odd compared to when r is even. Very recently it was realised by Takase [33] that Hill’s construction does not actually produce all the split regular representations. Furthermore, Takase gave a construction of all regular representations which correspond to conjugacy classes with separable characteristic polynomial mod \mathfrak{p} , assuming the residue characteristic p of \mathfrak{o} is not 2. At the same time, and independently, two general constructions of regular representations have been found. One is by Krakovski, Onn and Singla [18], which works whenever p is not 2, and the other is by Stasinski and Stevens [32]. The latter works for any \mathfrak{o} , and we therefore now have a complete construction of all the regular representations of $GL_N(\mathfrak{o}_r)$. This is currently the most general uniform construction of irreducible representations of $GL_N(\mathfrak{o}_r)$ available.

In Section 3 we give an introduction to the Clifford theory approach to the representations of $\mathrm{GL}_N(\mathfrak{o}_r)$. In Section 4 we define regular representations and give the construction when r is even. In Sections 5-7 we then focus on the various constructions of regular representations for r odd. Section 5 contains a summary of Hill's and Takase's constructions of regular semisimple representations. In Section 6 we give an outline of the construction of Krakovski, Onn and Singla. Finally, in Section 7 we elaborate on the main steps in the construction of Stasinski and Stevens. In the final Section 8 we mention some open problems.

Throughout the paper we have omitted most proofs, apart from the proofs of some occasional lemmas. On the other hand, we have tried to provide detailed explanations of many of the arguments.

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2. HISTORICAL OVERVIEW

The characters of $\mathrm{GL}_N(\mathfrak{o}_1) = \mathrm{GL}_N(\mathbb{F}_q)$ were determined in a classical paper of Green [9] and the representations can be constructed via Deligne-Lusztig theory. The representations of the finite groups $\mathrm{GL}_2(\mathfrak{o}_r)$, for all $r \geq 1$, have in one form or another been known to some mathematicians since the late 70s. There have been at least two different approaches to this problem. On the one hand, there is the Weil representation approach of Nobs and Wolfart (applied to the case $\mathfrak{o} = \mathbb{Z}_p$ in [23]). On the other hand, there is the approach via orbits and Clifford theory due to Kutzko (unpublished), and independently to Nagornyj [20]; see also [30].

The related case of $\mathrm{SL}_2(\mathbb{Z}_p)$, $p \neq 2$, was studied by Kloosterman [16, 17], Tanaka [34, 35], Kutzko (thesis; unpublished), and Shalika (for general \mathfrak{o} and $p \neq 2$) [27]. The Clifford theoretic approach of Kutzko and Shalika was rediscovered by Jaikin-Zapirain in [15, Section 7]. Another description of the representations of $\mathrm{SL}_2(\mathbb{Z}_p)$ (including the much more difficult case $p = 2$) was obtained by Nobs and Wolfart [22, 24] using Weil representations. The case $\mathrm{PGL}_2(\mathfrak{o})$, again with $p \neq 2$, was treated by Silberberger [29].

The representations of $\mathrm{GL}_3(\mathfrak{o})$ were studied by Nagornyj in [21], but the construction of representations was left incomplete. However, it was shown in [21] that the classification of representations of $\mathrm{GL}_N(\mathfrak{o})$ is a so-called wild problem, and in general one can therefore not expect an explicit and surveyable parametrisation of all the representations.

Recently, thorough and in-depth work on the representations of $\mathrm{GL}_3(\mathfrak{o})$ and $\mathrm{SL}_3(\mathfrak{o})$ (and related groups) has appeared in a series of papers by Avni, Klopsch, Onn and Voll; see, for example, [1, 2]. The results in [1] are based, among other things, on the Kirillov orbit method, which works for principal congruence subgroups of $\mathrm{SL}_3(\mathfrak{o})$ of index large enough compared to p , and only when \mathfrak{o} has characteristic 0. In [2] the authors employ Clifford theoretic methods to count the representations of $\mathrm{GL}_3(\mathfrak{o}_r)$ and $\mathrm{SL}_3(\mathfrak{o}_r)$ (when $\mathrm{char} \mathfrak{o} = 0$ or $\mathrm{char} \mathfrak{o} = p$ and p is large enough relative to r) and of $\mathrm{SL}_3(\mathfrak{o})$ (when $\mathrm{char} \mathfrak{o} = 0$ and p is large enough relative to the absolute ramification index of \mathfrak{o}). Analogous results are obtained for unitary groups corresponding to an unramified extension of \mathfrak{o} .

For the groups $GL_N(\mathfrak{o}_r)$ with $N \geq 2$, $r \geq 2$, the first general results seem to be due to Shintani in 1968 [28], who constructed the so-called *regular* representations when r is even. This construction was rediscovered in independent work of Hill around 1995 [12]. The series of papers by Hill [10, 11, 12, 13] use the method of orbits and Clifford theory to study and construct some of the representations of $GL_N(\mathfrak{o}_r)$. In particular, in addition to Shintani's construction of regular representations for r even, Hill went on to construct certain regular representations when r is odd. Over twenty years after Hill's work, it was realised by Takase [33] that Hill's construction of split regular representations is not exhaustive when the orbit is not semisimple. As mentioned in the introduction, recent constructions of regular representations have led to successively more general results, so that we now have a complete construction of all the regular representations of $GL_N(\mathfrak{o}_r)$. We will give a more detailed description of this work in subsequent sections.

Another approach to the representation theory of $GL_N(\mathfrak{o}_r)$ is based on viewing this group as the automorphism group of a rank N \mathfrak{o} -module. This was initiated by Onn [25], who defined a new type of induction functor, called infinitesimal induction, for general automorphism groups of \mathfrak{o} -modules of residual rank N . Infinitesimal induction complements the classical induction from parabolic subgroups, which in [25] is referred to as geometric induction. Decomposing these induced representations and using the known construction of (strongly) cuspidal representations for $GL_2(\mathfrak{o})$, leads to another classification of the representations of this group.

Finally, we mention a different approach to the representations of $GL_N(\mathfrak{o}_r)$, or more generally, for reductive groups over \mathfrak{o}_r . This approach is a cohomological construction of certain irreducible representations attached to characters of finite maximal tori. It was given by Lusztig in [19] in the case where \mathfrak{o} has positive characteristic, and for arbitrary \mathfrak{o} in [31]. This is a higher level generalisation of the classical construction of Deligne and Lusztig [6], which corresponds to the case $r = 1$. Another, "purely algebraic" (non-cohomological) construction of representations of certain split reductive groups, also attached to characters of finite maximal tori, was given by Gérardin [8]. In [19, Section 1] Lusztig suggested the problem of whether these representations are in fact the same as those given by the higher Deligne-Lusztig construction. This was recently answered in the affirmative for r even by Chen and Stasinski [5].

3. CLIFFORD THEORY FOR $GL_N(\mathfrak{o}_r)$

If G is a finite group, we will write $\text{Irr}(G)$ for the set of isomorphism classes of complex irreducible representations of G . For convenience, we will always consider an element $\rho \in \text{Irr}(G)$ as a representation, rather than an equivalence class of representations, that is, we identify $\rho \in \text{Irr}(G)$ with any representative of the isomorphism class ρ . One can view $\text{Irr}(G)$ as the set of irreducible characters of G , but we prefer to work with representations when possible. If G is abelian, we will often refer to a one-dimensional representation of G as a character. If $H \subseteq G$ is a subgroup and ρ is any representation of G we write $\rho|_H$ for the restriction of ρ to H .

Let \mathfrak{o} be a compact discrete valuation ring, that is, the ring of integers in a non-Archimedean local field with finite residue field, say \mathbb{F}_q , of characteristic p . Denote by \mathfrak{p} the maximal ideal of \mathfrak{o} , and by ϖ a fixed generator of \mathfrak{p} . For any integer $r \geq 1$ we write \mathfrak{o}_r for the finite ring $\mathfrak{o}/\mathfrak{p}^r$. We will also use \mathfrak{p} and ϖ to denote the

corresponding images of \mathfrak{p} and ϖ in \mathfrak{o}_r . Fix an integer $N \geq 2$ and, for any $r \geq 1$, put

$$\begin{aligned} G_r &= GL_N(\mathfrak{o}_r), \\ \mathfrak{g}_r &= M_N(\mathfrak{o}_r), \end{aligned}$$

where, for a commutative ring R , we use $M_N(R)$ to denote the algebra of $N \times N$ matrices over R . From now on, we consider a fixed $r \geq 2$. For any integer i such that $r \geq i \geq 1$, let $\rho_i = \rho_{r,i} : G_r \rightarrow G_i$ be the surjective homomorphism induced by the canonical map $\mathfrak{o}_r \rightarrow \mathfrak{o}_i$, and write $K^i = K_r^i = \text{Ker } \rho_i$. We also write ρ_i for the corresponding homomorphism $\mathfrak{g}_r \rightarrow \mathfrak{g}_i$. We thus have a descending chain of subgroups

$$G_r \supset K^1 \supset \cdots \supset K^r = \{1\},$$

where

$$K^i = 1 + \mathfrak{p}^i \mathfrak{g}_r.$$

With this description of the kernels, it is easy to show the commutator relation $[K^i, K^j] \subseteq K^{\min(i+j, r)}$, for $r \geq i, j \geq 1$. In particular, if $i \geq r/2$, then K_i is abelian, and if we let $l = \lceil \frac{r}{2} \rceil$, then K^l is the maximal abelian group among the kernels K^i . From now on, let $i \geq r/2$, that is, $i \geq l$. Then the map $x \mapsto 1 + \varpi^i x$ induces an isomorphism

$$(3.1) \quad \mathfrak{g}_{r-i} \xrightarrow{\sim} K^i.$$

The group G_r acts on \mathfrak{g}_{r-i} by conjugation, via its quotient G_{r-i} . This action is transformed by the above isomorphism into the action of G_r on its normal subgroup K^i . Let F be the fraction field of \mathfrak{o} . Fix an additive character $\psi : F \rightarrow \mathbb{C}^\times$ which is trivial on \mathfrak{o} but not on \mathfrak{p}^{-1} . For each $r \geq 1$ we can view ψ as a character of the group F/\mathfrak{p}^r whose kernel contains \mathfrak{o}_r . We will use ψ and the trace form $(x, y) \mapsto \text{tr}(xy)$ on \mathfrak{g}_r to set up a duality between the groups $\text{Irr}(K^i)$ and \mathfrak{g}_{r-i} . For $\beta \in M_N(\mathfrak{o}_r)$, define a homomorphism $\psi_\beta : K^i \rightarrow \mathbb{C}^\times$ by

$$(3.2) \quad \psi_\beta(1 + x) = \psi(\varpi^{-r} \text{tr}(\beta x)),$$

for $x \in \mathfrak{p}^i \mathfrak{g}_r$. Note that $\varpi^{-r} \text{tr}(\beta x)$ is a well defined element of F/\mathfrak{p}^r . Since ψ is trivial on \mathfrak{o}_r , ψ_β only depends on $x \bmod \mathfrak{p}^{r-i}$ (as it must in order to be well defined). Moreover, the map $\beta \mapsto \psi_\beta$ is a homomorphism whose kernel is $\mathfrak{p}^{r-i} \mathfrak{g}_r$, thanks to the non-degeneracy of the trace form. Hence it induces an isomorphism

$$\mathfrak{g}_r / \mathfrak{p}^{r-i} \mathfrak{g}_r \xrightarrow{\sim} \text{Irr}(K^i),$$

where we will usually identify $\mathfrak{g}_r / \mathfrak{p}^{r-i} \mathfrak{g}_r$ with \mathfrak{g}_{r-i} . For $g \in G_r$ we have

$$\psi_{g\beta g^{-1}}(x) = \psi(\varpi^{i-r} \text{tr}(g\beta g^{-1}x)) = \psi(\varpi^{i-r} \text{tr}(\beta g^{-1}xg)) = \psi_\beta(g^{-1}xg).$$

Thus the isomorphism (3.1) transforms the action of G_r on \mathfrak{g}_r into (the inverse) conjugation of characters.

Remark 3.1. In the above we have used adjoint orbits (i.e., conjugacy classes) in $\mathfrak{g}_r / \mathfrak{p}^{r-i} \mathfrak{g}_r$ to parametrise orbits of characters of K^i . From some points of view it is more natural to use co-adjoint orbits in the dual

$$\mathfrak{g}_r^* := \text{Hom}_{\mathfrak{o}_r}(\mathfrak{g}_r, \mathfrak{o}_r).$$

Indeed the pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}_r^* \times \mathfrak{g}_r \rightarrow \mathfrak{o}_r$ given by $\langle f, \beta \rangle = f(\beta)$ is non-degenerate and one can define

$$\psi_f(1 + x) = \psi(\varpi^{-r} \langle f, x \rangle),$$

where $f \in \mathfrak{g}_r^*$. This induces an isomorphism $\mathfrak{g}_r^*/\mathfrak{p}^{r-i}\mathfrak{g}_r^* \cong \mathfrak{g}_{r-i}^* \xrightarrow{\sim} \text{Irr}(K^i)$, and has the advantage of generalising to Chevalley groups other than GL_N (where the trace form may be degenerate); see [8]. However, for GL_N we prefer to work with elements in \mathfrak{g}_r rather than elements in its dual, and we can translate between the two by means of the G_r -equivariant bijection induced by the trace form.

If G is a finite group, $H \subseteq G$ is a subgroup and $\rho \in \text{Irr}(H)$, we will write $\text{Irr}(G \mid \rho)$ for the set of $\pi \in \text{Irr}(G)$ such that π contains ρ on restriction to H , that is,

$$\text{Irr}(G \mid \rho) = \{\pi \in \text{Irr}(G) \mid \langle \pi|_H, \rho \rangle \neq 0\}.$$

Moreover, if N is a normal subgroup of G , then G acts on $\text{Irr}(N)$ by $\rho \mapsto {}^g\rho$, where ${}^g\rho(n) := \rho(gng^{-1})$, for $g \in G$, $n \in N$. In this case, we define the *stabiliser* of $\rho \in \text{Irr}(N)$ to be $G(\rho) = \{g \in G \mid {}^g\rho \cong \rho\}$. We will subsequently make use of the following well known results from Clifford theory of finite groups:

Theorem 3.2. *Let G be a finite group, and N a normal subgroup. Then the following hold:*

- (i) (*Clifford's theorem*) *If $\pi \in \text{Irr}(G)$, then $\pi|_N = e \bigoplus_{\rho \in \Omega} \rho$, where $\Omega \subseteq \text{Irr}(N)$ is an orbit under the action of G on $\text{Irr}(N)$ by conjugation, and e is a positive integer.*
- (ii) *Suppose that $\rho \in \text{Irr}(N)$. Then $\theta \mapsto \text{Ind}_{G(\rho)}^G \theta$ is a bijection from $\text{Irr}(G(\rho) \mid \rho)$ to $\text{Irr}(G \mid \rho)$.*
- (iii) *Let H be a subgroup of G containing N , and suppose that $\rho \in \text{Irr}(N)$ has an extension $\tilde{\rho}$ to H (i.e., $\tilde{\rho}|_N = \rho$). Then*

$$\text{Ind}_N^H \rho = \bigoplus_{\chi \in \text{Irr}(H/N)} \tilde{\rho}\chi,$$

where each $\tilde{\rho}\chi$ is irreducible, and where we have identified $\text{Irr}(H/N)$ with $\{\chi \in \text{Irr}(H) \mid \chi(N) = 1\}$.

For proofs of the above, see for example [14], 6.2, 6.11 and 6.17, respectively. The above results (i) and (ii) show that in order to obtain a classification of the representations of G_r , it is enough to classify the orbits of characters ψ_β of a normal subgroup K^i , and to construct all the elements in $\text{Irr}(G_r(\psi_\beta) \mid \psi_\beta)$, that is, to decompose $\text{Ind}_{K^i}^{G_r(\psi_\beta)} \psi_\beta$ into irreducible representations. This is what we shall do in the following, taking $K^i = K^l$.

Remark 3.3. By an (algebraic) construction of some irreducible representations (or characters) of G_r via Clifford theory, we will always mean a general (i.e., valid for all G_r) finite sequence of extensions and inductions of characters, starting from the one-dimensional characters of K^l . Note that the existence of an extension of a representations is allowed to be a non-constructive fact.

In order to have a complete understanding of representations constructed via Clifford theory, it is necessary to have an understanding of the G_r conjugacy classes (or orbits) in \mathfrak{g}_r , because $\text{Irr}(G_r(\psi_\beta) \mid \psi_\beta) = \text{Irr}(G_r(\psi_{\beta'}) \mid \psi_{\beta'})$ if β and β' are conjugate. One cannot expect to have an explicit understanding of all the orbits, but we *do* have an explicit normal form for regular orbits, as we will see next.

4. REGULAR REPRESENTATIONS, r EVEN

An irreducible representation π of G_r is called *regular* if $\pi|_{K^l}$ contains ψ_β with $\beta \in \mathfrak{g}_r$ regular. By a result of Hill [12, Theorem 3.6] $\beta \in \mathfrak{g}_r$ is regular if and only if its image $\bar{\beta} \in \mathfrak{g}_1 = M_N(\mathbb{F}_q)$ is regular, that is, if $\dim C_{\mathfrak{g}_1}(\bar{\beta}) = N$. There are several equivalent characterisations of regular elements in \mathfrak{g}_1 ; in particular, $\bar{\beta} \in \mathfrak{g}_1$ is regular iff $C_{\mathfrak{g}_1}(\bar{\beta})$ is abelian iff the characteristic polynomial of $\bar{\beta}$ equals the minimal polynomial iff $\bar{\beta}$ is conjugate to a companion matrix. Note that β depends on the choice of ψ , but for any other choice ψ' we have $\psi_\beta = \psi'_{a\beta}$, for some $a \in \mathfrak{o}_r^\times$, and since β is regular if and only if $a\beta$ is regular, regularity is an intrinsic property of a representation $\pi \in \text{Irr}(G_r)$.

There are three special properties of regular elements which will allow us to construct and completely classify all the regular representations:

- (i) We can tell explicitly when two regular elements are G_r -conjugate, namely, if and only if their companion matrices coincide.
- (ii) The centraliser $C_{G_r}(\beta)$ of a regular element $\beta \in \mathfrak{g}_r$ is abelian.
- (iii) For any $1 \leq s \leq r$, the map $\rho_s : C_{G_r}(\beta) \rightarrow C_{G_s}(\beta_s)$ is surjective, where β_s is the image of β under $\rho_s : \mathfrak{g} \rightarrow \mathfrak{g}_s$.

We will illustrate this in the construction of all regular representations of G_r when r is even, given below.

Remark 4.1. If $\pi \in \text{Irr}(G_r \mid \psi_\beta)$, then (3.2) implies that π has K^{r-1} in its kernel if and only if $\bar{\beta} = 0$. Thus π factors through G_{r-1} if and only if $\bar{\beta} = 0$. If this is the case, π is called *imprimitive*. If π does not factor through G_{r-1} it is called *primitive*. Note that a regular representation is necessarily primitive. On the other hand, there exist irreducible representations of G_r which are not regular, because they factor through G_{r-1} , but are regular when viewed as characters of G_{r-1} . For example, take the representations of $GL_2(\mathfrak{o}_4)$ with $\beta = \begin{pmatrix} 0 & \pi \\ 0 & 0 \end{pmatrix}$.

From now on, let $\psi_\beta \in \text{Irr}(K^l)$ with $\beta \in \mathfrak{g}_r$ regular. Let $l' = r - l$, so that $l = l'$ when r is even and $l' = l - 1$ when r is odd. As indicated in the previous section, the stabiliser $G_r(\psi_\beta)$ plays an important role in the construction of representations of G_r . The formula $\psi_\beta(g^{-1}xg) = \psi_{g\beta g^{-1}}(x)$, together with the fact that $\psi_\beta = \psi_{\beta'} \Leftrightarrow \beta \equiv \beta' \pmod{\mathfrak{p}^{l'}}$, implies that

$$(4.1) \quad G_r(\psi_\beta) = C_{G_r}(\beta + \mathfrak{p}^{l'} \mathfrak{g}_r).$$

An important corollary of [12, Theorem 3.6] is that for regular β , and any s such that $r \geq s \geq 1$, the natural reduction map

$$C_{G_r}(\beta) \longrightarrow C_{G_s}(\beta_s)$$

is surjective. Another corollary of [12, Theorem 3.6] is that for regular β we have $C_{G_r}(\beta) = \mathfrak{o}_r[\beta]^\times$, so that in particular, the centraliser is abelian. Together with (4.1) these two results imply that

$$(4.2) \quad G_r(\psi_\beta) = C_{G_r}(\beta)K^{l'} = \mathfrak{o}_r[\beta]^\times K^{l'}.$$

We now give the construction of regular representations of G_r in the case when r is even. Suppose that r is even, so that $l = l'$. Let $\theta \in \text{Irr}(C_{G_r}(\beta))$ be any irreducible component of $\text{Ind}_{C_{G_r}(\beta) \cap K^l}^{C_{G_r}(\beta)}(\psi_\beta|_{C_{G_r}(\beta) \cap K^l})$. Since $C_{G_r}(\beta)$ is abelian θ is

one-dimensional, and hence it agrees with ψ_β on $C_{G_r}(\beta) \cap K^l$. It is then easy to check that

$$\tilde{\psi}_\beta(ck) := \theta(c)\psi_\beta(k)$$

is a well defined one-dimensional representation of $G_r(\psi_\beta)$, and by construction it is an extension of ψ_β . By Theorem 3.2 (iii) we obtain

$$\text{Irr}(G_r(\psi_\beta) \mid \psi_\beta) = \{\tilde{\psi}_\beta\chi \mid \chi \in \text{Irr}(C_{G_l}(\beta_l))\},$$

where $\beta_l \in \mathfrak{g}_l$ is the image of β . Hence Theorem 3.2 (ii) implies that there is a bijection

$$\begin{aligned} \text{Irr}(C_{G_l}(\beta_l)) &\longrightarrow \text{Irr}(G_r \mid \psi_\beta) \\ \chi &\longmapsto \text{Ind}_{G_r(\psi_\beta)}^{G_r} \tilde{\psi}_\beta\chi. \end{aligned}$$

Note that this is not canonical, but depends on the choice of $\tilde{\psi}_\beta$. We have thus constructed the irreducible representations of G_r containing ψ_β , in terms of the irreducible representations of the abelian group $C_{G_l}(\beta_l)$ (which we consider known; cf. Remark 3.3). Note that if we start with another element in the conjugacy class of β , we obtain the same set of irreducible representations of G_r . Thus, when r is even, running through a set of representatives for the regular conjugacy classes in \mathfrak{g}_l , yields all the regular representations of G_r exactly once.

As far as the author is aware, the above construction is due to Shintani [28, §2, Theorem 2], although Shintani does not prove that every regular element in \mathfrak{g}_l is regular mod \mathfrak{p} . The construction was rediscovered by Hill [12, Theorem 4.1].

It remains to construct the regular representations of G_r when r is odd. This requires additional methods, due to the fact that $G_r(\psi_\beta) = C_{G_r}(\beta)K^{l'}$, and it is not possible to extend ψ_β from K^l to $G_r(\psi_\beta)$. Instead, one has to take several intermediate steps consisting of extensions and inductions. In the following we will give an exposition of the currently known constructions (sometimes partial) of regular representations of G_r , for r odd.

5. THE CONSTRUCTIONS OF HILL AND TAKASE

From now on and until the end of Section 7 we will assume that r is odd, so that $l' := r - l = l - 1$. In this case, Hill [12] claimed to give a construction of so-called *split* regular representations, that is, those for which the characteristic polynomial of $\bar{\beta} \in \mathfrak{g}_1$ splits into linear factors over \mathbb{F}_q . Takase [33] recently pointed out a gap in the proof of Hill's result [12, Theorem 4.6] and proved that the construction exhausts at most the split regular *semisimple* representations, but does not exhaust all split regular representations. We give a summary of Hill's construction following [12], point out two problems in the proof, and state the correction/generalisation due to Takase.

We have an isomorphism $K^{l'}/K^l \cong \mathfrak{g}_1$, and we can identify any subgroup of $K^{l'}$ which contains K^l with a sub-vectorspace of \mathfrak{g}_1 . Define the alternating bilinear form

$$\begin{aligned} B_\beta : K^{l'}/K^l \times K^{l'}/K^l &\longrightarrow \mathbb{F}_q \\ B_\beta((1 + \pi^{l'}x)K^l, (1 + \pi^{l'}y)K^l) &= \text{tr}(\bar{\beta}(\bar{x}\bar{y} - \bar{y}\bar{x})), \end{aligned}$$

where the bars denote reductions mod \mathfrak{p} . The following is [12, Lemma 4.5], rewritten in our notation.

Lemma 5.1. *Suppose that $\beta \in \mathfrak{g}_r$ is split regular. Then there exists a subgroup H_β of $K^{l'}$ such that H_β contains K^l and such that H_β/K^l is a maximal isotropic subspace of $K^{l'}/K^l$ with respect to the form B_β . Moreover, H_β is a normal subgroup of $G_r(\psi_\beta)$.*

We recall that a subspace U of a vector space V with a bilinear form $B(\cdot, \cdot)$ is called *isotropic* (or sometimes *totally isotropic*) if $U \subseteq U^\perp$, that is, if $B(U, U) = 0$. Furthermore, U is called *maximal isotropic* (or sometimes *Lagrangian*) if it is not properly contained in any isotropic subspace, or equivalently, if $U = U^\perp$.

The proof of the above lemma consists of taking $H_\beta = (B \cap K^{l'})K^l$, where B is the upper-triangular subgroup of G_r , and showing that it has the required properties, using the assumption that $\bar{\beta}$ is upper-triangular. Thus in Hill's construction, H_β is in fact independent of β .

Hill's main theorem [12, Theorem 4.6] regarding the construction of split regular representations for r odd claims that if $\beta \in \mathfrak{g}_r$ is split regular, then for every $\pi \in \text{Irr}(G_r \mid \psi_\beta)$, there exists a subgroup H_β as in Lemma 5.1 and an extension $\tilde{\psi}_\beta$ of ψ_β to $C_{G_r}(\beta)H_\beta$ such that

$$\pi = \text{Ind}_{C_{G_r}(\beta)H_\beta}^{G_r} \tilde{\psi}_\beta.$$

Unfortunately, Hill's proof of [12, Theorem 4.6] suffers from two problems. One is that a certain counting argument only goes through when $\bar{\beta}$ is assumed to be semisimple (see [33, Proposition 2.1.1]), so that Hill's construction does not exhaust the split regular representations. The other problem is that, in the second paragraph of the proof, it is asserted that a result of Brauer implies that the number of $C_{G_r}(\beta)H_\beta/N$ -stable characters of H_β/N is equal to the number of $C_{G_r}(\beta)H_\beta/N$ -stable conjugacy classes of H_β/N , where $N = \text{Ker } \psi_\beta$. However, the quoted result of Brauer holds only for characters/conjugacy classes fixed by a single element in a group, and does not necessarily apply to the whole group $C_{G_r}(\beta)H_\beta/N$. We remark that by results of Glauberman and Isaacs (see [14, (13.24)] the appropriate generalisation of Brauer's result holds for coprime group actions, but may fail otherwise. Since p divides the orders of both $C_{G_r}(\beta)H_\beta/N$ and H_β/N , the crucial step in Hill's proof which asserts the existence of an extension of ψ_β to $C_{G_r}(\beta)H_\beta$ remains unclear.

In addition to the split regular representations, there are many regular representations which are not split, in particular the *cuspidal* representations, that is, those where $\bar{\beta}$ has irreducible characteristic polynomial. In [13] Hill gave a construction of so-called *strongly semisimple* representations, that is, those for which $\bar{\beta}$ is semisimple and $\beta_{l'} \in \mathfrak{g}_{l'}$ has additive Jordan decomposition $\beta_{l'} = s + n$, with n in the centre of the algebra $C_{\mathfrak{g}_{l'}}(s)$.

Example 5.2. Consider the function $\iota : \mathbb{F}_q \rightarrow \mathfrak{o}_r$, which is the multiplicative section extended by setting $\iota(0) = 0$. This induces an injective function $\mathfrak{g}_1 \rightarrow \mathfrak{g}_r$. An element in \mathfrak{g}_r is called *semisimple* if it is the image of a semisimple element in \mathfrak{g}_1 under the map $\mathfrak{g}_1 \rightarrow \mathfrak{g}_r$. Then any $\beta \in \mathfrak{g}_r$ has a unique Jordan decomposition $\beta = s + n$, where s is semisimple, n is nilpotent and $sn = ns$ (see [13, Proposition 2.3]). If $n = 0$, then β is strongly semisimple, so in particular, there are strongly semisimple

representations which are not regular. The strongly semisimple representations include the cuspidal ones (see [13, Proposition 4.4]).

Hill's construction of strongly semisimple representations for r odd is summarised in the following (cf. [13, Proposition 3.6]) result:

Theorem 5.3. *Let $\pi \in \text{Irr}(G_r \mid \psi_\beta)$ be strongly semisimple. Then there exists a $\rho \in \text{Irr}(K^{l'} \mid \psi_\beta)$ and an extension $\tilde{\rho}$ of ρ to $G_r(\psi_\beta)$ such that*

$$\pi = \text{Ind}_{G_r(\psi_\beta)}^{G_r} \tilde{\rho}.$$

Note that the only non-trivial part of this theorem is that ρ has an extension. In fact, it follows from the proof in [13] that *every* $\rho \in \text{Irr}(K^{l'} \mid \psi_\beta)$ extends to $G_r(\psi_\beta)$. Moreover, by Theorem 3.2 (ii), distinct extensions of ρ give rise to distinct representations π .

The elements of $\text{Irr}(K^{l'} \mid \psi_\beta)$ are constructed in [12, Proposition 4.2 (3)], so that together with the above theorem, this gives a complete construction of strongly semisimple representations, up to a knowledge of the elements in $\text{Irr}(G_r(\psi_\beta)/K^{l'}) \cong \text{Irr}(C_{G_{l'}}(\beta_{l'}))$. A version of Theorem 5.3 holds also when r is even; see [13, Proposition 3.3].

We see that out of the regular representations, Hill's constructions cover at most those which are semisimple (i.e., where $\bar{\beta}$ is semisimple). The next step was taken recently by Takase, who proved the following (see [33, Theorem 3.2.2, 5.2.1 and 5.3.1]):

Theorem 5.4. *Let $\pi \in \text{Irr}(G_r \mid \psi_\beta)$ be a regular character and suppose that $\bar{\beta}$ satisfies at least one of the following properties:*

- (i) $\bar{\beta}$ has separable characteristic polynomial and $p > 2$,
- (ii) $\bar{\beta}$ has Jordan blocks of size at most 4 and $p > 7$.

Then there exists a $\rho \in \text{Irr}(K^{l'} \mid \psi_\beta)$ and an extension $\tilde{\rho}$ of ρ to $G_r(\psi_\beta)$ such that $\pi = \text{Ind}_{G_r(\psi_\beta)}^{G_r} \tilde{\rho}$.

Just as for Hill's theorem on strongly semisimple representations above, the difficulty in Takase's proof lies in showing that every $\rho \in \text{Irr}(K^{l'} \mid \psi_\beta)$ extends to $G_r(\psi_\beta)$. The existence of an extension follows from the vanishing of the cohomology group $H^2(\mathbb{F}_q[\bar{\beta}]^\times, \mathbb{C}^\times)$, the so-called Schur multiplier. When $\bar{\beta}$ has irreducible characteristic polynomial, $\mathbb{F}_q[\bar{\beta}]$ is a finite field, so $\mathbb{F}_q[\bar{\beta}]^\times$ is cyclic. In this case it is well known that $H^2(\mathbb{F}_q[\bar{\beta}]^\times, \mathbb{C}^\times)$ is trivial. For $p > 2$ Takase reduces the separable case to the irreducible, and thus proves Theorem 5.4 when $\bar{\beta}$ satisfies the first condition; cf. [33, Theorem 4.3.2]. The existence of an extension when $\bar{\beta}$ satisfies the second condition is proved in [33] by explicit computation of the relevant cocycles.

These results led Takase to conjecture that a certain element in the Schur multiplier is always trivial for p large enough; see [33, Conjecture 4.6.5].

6. THE CONSTRUCTION OF KRAKOVSKI, ONN AND SINGLA

We will now describe the construction of regular representations of G_r , r odd, due to Krakovski, Onn and Singla [18]. This gives a construction of all the regular representations, provided the residue characteristic p of \mathfrak{o} is *odd*. Furthermore, [18] also contains constructions and enumeration of all the regular representations

of $\mathrm{SL}_N(\mathfrak{o}_r)$ when $p > N$, and of the unitary groups $\mathrm{SU}_N(\mathfrak{o}_r)$ and $\mathrm{GU}_N(\mathfrak{o}_r)$ with respect to a quadratic unramified extension of \mathfrak{o} (with some restrictions on p). The construction in [18] was inspired by a construction of Jaikin-Zapirain for $\mathrm{SL}_2(\mathfrak{o}_r)$, $p > 2$; see [15, Section 7]. We continue to assume that $r = l + l'$ is odd. The following result is [18, Theorem 3.1], which is a more detailed statement of [18, Theorem A]. We state this only for GL_N , in a form slightly adapted to our present notation.

Theorem 6.1. *Assume that \mathfrak{o} has residue characteristic $p > 2$. Let $\sigma \in \mathrm{Irr}(K^{l'} \mid \psi_\beta)$ with β regular. Then σ has an extension $\tilde{\sigma}$ to $G_r(\psi_\beta)$, and thus any $\pi \in \mathrm{Irr}(G_r \mid \psi_\beta)$ is of the form $\pi = \mathrm{Ind}_{G_r(\psi_\beta)}^{G_r} \tilde{\sigma}$, for some extension $\tilde{\sigma}$.*

In particular, this proves a strengthened form of Takase's conjecture mentioned above, namely for all $p > 2$ (another proof of this, for all p , follows from the construction of Stasinski and Stevens). We elaborate on the proof of [18, Theorem 3.1] in order to provide some of the details of the construction. As we have already remarked in previous sections, the main difficulty is to show that every $\sigma \in \mathrm{Irr}(K^{l'} \mid \psi_\beta)$ extends to $G_r(\psi_\beta)$. We will mainly formulate things in our present notation, but use the notation of [18] where possible.

6.1. Characters. Assume that $p > 2$. For i such that $r/2 \leq i < r$, the exponential map $\exp : x \mapsto 1 + x$ gives an isomorphism $\mathfrak{p}^i \mathfrak{g}_r \rightarrow K^i$ (we already saw this in Section 3, and it works for any p). Moreover, when $p > 2$ and $r/3 \leq i < r/2$, the exponential map $\exp : x \mapsto 1 + x + \frac{1}{2}x^2$ gives a bijection $\mathfrak{p}^i \mathfrak{g}_r \rightarrow K^i$, which is however not an isomorphism in general. As usual, the inverse of this exponential map is given by the logarithm $\log : 1 + x \mapsto x - \frac{1}{2}x^2$. Every $\beta \in \mathfrak{g}_r$ defines a character

$$\varphi_\beta : \mathfrak{g}_r \longrightarrow \mathbb{C}^\times, \quad \text{where } \varphi_\beta(x) = \psi(\varpi^{-r} \mathrm{tr}(\beta x)).$$

The corresponding map $\beta \mapsto \varphi_\beta$ is an isomorphism. Any $\theta \in \mathrm{Irr}(\mathfrak{p}^l \mathfrak{g}_r)$ can be precomposed with the logarithm map $\log : K^l \rightarrow \mathfrak{p}^l \mathfrak{g}_r$, $1 + x \mapsto x$, to give a character $\log^* \theta := \theta \circ \log \in \mathrm{Irr}(K^l)$, such that, for $1 + x \in K^l$,

$$(\log^* \theta)(1 + x) = \varphi_\beta(x),$$

where β is determined by θ . Note that $\log^* \theta = \psi_\beta$, where ψ_β is as in (3.2). In particular, φ_β restricts to θ on $\mathfrak{p}^l \mathfrak{g}_r$, but for a given θ , there is more than one β such that φ_β restricts to θ , since the restriction only depends on $\beta \bmod \mathfrak{p}^l$.

A crucial step in [18] (due to Jaikin-Zapirain for SL_2), is to extend the above definition of $\log^* \theta$, in order to give a useful description of certain characters on any subgroup $K^l \subseteq J_\beta \subseteq K^{l'}$ such that J_β/K^l is a maximal isotropic subspace for the form B_β defined in Section 5. This is the motivation behind [18, Lemma 3.2], and the essential reason for the assumption $p > 2$. The following result gives a summary of the key facts involved (see [18, Lemma 3.2 and Section 3.2]).

Lemma 6.2. *Let J_β be such that J_β/K^l is a maximal isotropic subspace. Let θ'' be the restriction of a character $\varphi_\beta \in \mathrm{Irr}(\mathfrak{g}_r)$ to $\mathfrak{p}^{l'} \mathfrak{g}_r$. Then the function $\log^* \theta'' : K^{l'} \rightarrow \mathbb{C}^\times$ defines a multiplicative character when restricted to J_β .*

Proof. Let $1 + \varpi^{l'} x$ and $1 + \varpi^{l'} y$ be elements in J_β . Direct computation yields the commutator

$$g := [(1 + \varpi^{l'} x), (1 + \varpi^{l'} y)] = 1 + \varpi^{2l'} (xy - yx).$$

Since we are assuming that $p > 2$, we have a unique square root $g^{1/2} = 1 + \frac{1}{2}\varpi^{2l'}(xy - yx)$. In particular, since $2l' = r - 1$, $g^{1/2}$ is in the centre of $K^{l'}$. Thus,

$$\begin{aligned} \log((1 + \varpi^{l'}x)(1 + \varpi^{l'}y)) &= \log((1 + \varpi^{l'}x)(1 + \varpi^{l'}y)g^{-1/2}g^{1/2}) \\ &= \log((1 + \varpi^{l'}x)(1 + \varpi^{l'}y)g^{-1/2}) + \log(g^{1/2}) \\ &= \log(1 + \varpi^{l'}(x + y) + \frac{1}{2}\varpi^{2l'}(xy + yx)) + \log(g^{1/2}) \\ &= \log(1 + \varpi^{l'}x) + \log(1 + \varpi^{l'}y) + \log(g^{1/2}), \end{aligned}$$

where the second equality follows from the fact that $g^{1/2}$ is central. Applying θ'' , we get

$$\begin{aligned} \theta''(\log((1 + \varpi^{l'}x)(1 + \varpi^{l'}y))) &= \theta''(\log(1 + \varpi^{l'}x)) + \theta''(\log(1 + \varpi^{l'}y)) \\ &\quad + \theta''(\log(g^{1/2})) \\ &= \theta''(\log(1 + \varpi^{l'}x)) + \theta''(\log(1 + \varpi^{l'}y)) \\ &\quad + \psi\left(\frac{1}{2}\varpi^{-1} \operatorname{tr}(\beta(xy - yx))\right) \\ &= \theta''(\log(1 + \varpi^{l'}x)) + \theta''(\log(1 + \varpi^{l'}y)), \end{aligned}$$

where the last equality follows from the fact that $\operatorname{tr}(\bar{\beta}(\bar{x}\bar{y} - \bar{y}\bar{x})) = B_\beta(1 + \varpi^{l'}x, 1 + \varpi^{l'}y) = 0$, since J_β/K^l is isotropic. \square

The crucial corollary of this lemma is that any $\log^* \theta \in \operatorname{Irr}(K^l)$ extends to J_β by the same formula, that is, $\log^* \theta'' = \log^* \varphi_\beta$. We emphasise that the key is not just that $\log^* \theta$ has an extension to J_β (this is true for any p , by [12, Proposition 4.2]), but that there is an extension given by an explicit formula which makes it evident that the extension is stabilised by any $g \in C_{G_r}(\beta)$ which normalises J_β . As we will explain below, the p -Sylow subgroup P_β of $C_G(\beta)$ normalises J_β , so the extension $\log^* \theta''$ of $\log^* \theta$ to J_β is stabilised by P_β . Note that it is not known whether all of $C_G(\beta)$ normalises J_β , in general.

We now describe the representations of the non-abelian group $K^{l'}$, following Hill [12, Proposition 4.2]. It is easy to check that the radical of the bilinear form B_β introduced in Section (5), is $(C_{G_r}(\beta) \cap K^{l'})K^l/K^l$. There is then a subgroup $K^l \subseteq J_\beta \subseteq K^{l'}$ such that J_β/K^l is a maximal isotropic subspace. The radical and maximal isotropic subspace correspond to two subspaces of $M_N(\mathbb{F}_q) \cong K^{l'}/K^l$, and we let

$$\mathfrak{r}_\beta \quad \text{and} \quad \mathfrak{j}_\beta$$

denote the inverse images in $\mathfrak{p}^{l'}\mathfrak{g}_r$ of these two subspaces, respectively, under the map $\mathfrak{p}^{l'}\mathfrak{g}_r \rightarrow M_N(\mathbb{F}_q)$, $\varpi^{l'}x \mapsto \bar{x}$. Clearly \mathfrak{r}_β and \mathfrak{j}_β only depend on $\bar{\beta} \in \mathfrak{g}_r$. Let $\theta \in \operatorname{Irr}(\mathfrak{p}^l\mathfrak{g}_r)$, and let θ' be an extension of θ to \mathfrak{r}_β (here we are just talking about characters of abelian groups). Then θ' determines a unique irreducible representation of $K^{l'}$, which arises as follows. Let θ'' be an extension of θ' to \mathfrak{j}_β . Then $\log^* \theta''$ is a character of the group J_β thanks to Lemma 6.2, and

$$\operatorname{Ind}_{J_\beta}^{K^{l'}}(\log^* \theta'')$$

can be shown to be irreducible. In fact, it is the unique element in $\operatorname{Irr}(K^{l'} \mid \log^* \theta')$.

6.2. Construction of representations. From now on, let $\theta \in \text{Irr}(\mathfrak{p}^l \mathfrak{g}_r)$ be a character that corresponds to a regular element, that is $\log^* \theta = \psi_\beta$, where $\beta \in \mathfrak{g}_r$ is regular (recall that ψ_β only depends on the coset $\beta + \mathfrak{p}^{l'} \mathfrak{g}_r$).

Lemma 6.3. *Let $\sigma \in \text{Irr}(K^{l'} \mid \log^* \theta)$. Then $G_r(\sigma) = G_r(\psi_\beta)$.*

Proof. Let $\theta' \in \text{Irr}(\mathfrak{r}_\beta)$ be the unique extension of θ that corresponds to σ . Choose $\beta' \in \mathfrak{g}_r$ such that $\varphi_{\beta'} \in \text{Irr}(\mathfrak{g}_r)$ is an extension of θ' . Then $\varphi_{\beta'}$ is also an extension of θ , so $\beta' \equiv \beta \pmod{\mathfrak{p}^{l'} \mathfrak{g}_r}$, and by (4.2) we have

$$G_r(\sigma) \subseteq G_r(\log^* \theta) = C_{G_r}(\beta) K^{l'} = C_{G_r}(\beta') K^{l'},$$

where the first inclusion follows from the fact that $\log^* \theta$ is the unique irreducible character of K^l contained in σ (the orbit of the restriction of σ to K^l consists of copies of ψ_β since $K^{l'}$ stabilises ψ_β).

For the reverse inclusion, note that $C_{G_r}(\beta')$ stabilises $\varphi_{\beta'}$, hence its restriction θ' , and hence the character $\log^* \theta'$. Since σ is the unique representation in $\text{Irr}(K^{l'} \mid \log^* \theta')$, σ is stabilised by $C_{G_r}(\beta')$, and so $C_{G_r}(\beta') K^{l'} \subseteq G_r(\sigma)$. \square

We now explain how to show that σ extends to the stabiliser $G_r(\psi_\beta)$. For this, it will be enough (by Lemma 6.3 and [14, Corollary 11.31]) to show that σ extends to the p -Sylow subgroup of $G_r(\psi_\beta)$ (which is unique since $G_r(\psi_\beta)$ is abelian modulo the p -group $K^{l'}$). Let P_β denote the p -Sylow subgroup of $C_{G_r}(\beta)$. The following crucial lemma, see [18, Lemma 3.4], goes back to Howe:

Lemma 6.4. *Let V be a finite dimensional \mathbb{F}_p -vector space and α an antisymmetric bilinear form on V . Suppose that P is a p -group which acts on V and preserves α . Then there exists a maximal isotropic subspace U of V which is P -invariant.*

The group P_β acts on $K^{l'}$ and K^l by conjugation, and hence induces an action on the vector space $K^{l'}/K^l$. By the above lemma, there exists a maximal isotropic subspace of $K^{l'}/K^l$ which is stable under this action of P_β , that is, there is a subgroup $K^l \subseteq J_\beta \subseteq K^{l'}$, such that the image of J_β in $K^{l'}/K^l$ is a maximal isotropic subspace and such that J_β is normalised by P_β . As in the proof of Lemma 6.3, let $\theta' \in \text{Irr}(\mathfrak{r}_\beta)$ be the unique extension of θ that corresponds to σ and $\varphi_\beta \in \text{Irr}(\mathfrak{g}_r)$ an extension of θ' . Then the restriction $\varphi_\beta|_{J_\beta}$ is stabilised by P_β (because it is stabilised by all of $C_{G_r}(\beta)$), and thus

$$\log^*(\varphi_\beta|_{J_\beta})$$

is a character of J_β (by Lemma 6.2), which is stabilised by P_β . Here we again see the crucial role played by Lemma 6.2 as well as the order in which choices are made: For any $\sigma \in \text{Irr}(K^{l'} \mid \psi_\beta)$, there is a unique $\theta' \in \text{Irr}(\mathfrak{r}_\beta)$, and this extends to a $\theta'' \in \text{Irr}(J_\beta)$ such that $\log^* \theta'' \in \text{Irr}(J_\beta)$ is stabilised by $C_{G_r}(\beta)$.

Since $\log^*(\varphi_\beta|_{J_\beta})$ is one-dimensional and P_β is abelian, this character extends to a character $\omega \in \text{Irr}(P_\beta J_\beta)$. The induced representation

$$\sigma' := \text{Ind}_{P_\beta J_\beta}^{P_\beta K^{l'}} \omega$$

has dimension

$$[P_\beta K^{l'} : P_\beta J_\beta] = \frac{|P_\beta| \cdot |K^{l'}| / |P_\beta \cap K^{l'}|}{|P_\beta| \cdot |J_\beta| / |P_\beta \cap J_\beta|} = \frac{|P_\beta \cap J_\beta|}{|P_\beta \cap K^{l'}|} [K^{l'} : J_\beta].$$

Since J_β contains the group $(C_{G_r}(\beta) \cap K^{l'})K^l$ (since every maximal isotropic subspace contains the radical of the form), we have $P_\beta \cap J_\beta \supseteq P_\beta \cap K^{l'}$. The reverse inclusion is trivial, so we have $\dim \sigma' = [K^{l'} : J_\beta] = \dim \sigma$. Since σ' must contain σ on restriction to $K^{l'}$ (because σ' contains $\log^* \theta'$), σ' must be an extension of σ (so in particular, σ' must be irreducible). Thus σ extends to the p -Sylow in $G_r(\psi_\beta)$ and hence to all of $G_r(\psi_\beta)$, by the above remarks. This concludes the proof of Theorem 6.1.

7. THE CONSTRUCTION OF STASINSKI AND STEVENS

In this section we summarise forthcoming work of Stasinski and Stevens [32] which gives a construction of all the regular representations of $G_r = GL_N(\mathfrak{o}_r)$, without any restriction on the residue characteristic. As in the previous two sections, we assume that $r = l + l'$ is odd.

One of the key distinguishing features of the present approach is the systematic use of the subgroup structure of G_r provided by lattice chains. In particular, for a given regular orbit, two specific associated parahoric subgroups and their filtrations will play a crucial role. The construction is somewhat analogous to the construction of supercuspidal representations of Bushnell and Kutzko [4], but with the difference that for us everything takes place inside G_r and all relevant centralisers are abelian (because we consider only regular representations).

7.1. Subgroup structure. Let $\mathfrak{A} \subseteq \mathfrak{g}_r = M_N(\mathfrak{o}_r)$ be a parahoric subalgebra, that is, the preimage under the reduction mod \mathfrak{p} map of a parabolic subalgebra of $\mathfrak{g}_1 = M_N(\mathbb{F}_q)$. Let \mathfrak{P} denote the preimage of the corresponding nilpotent radical of the parabolic subalgebra. A parabolic subalgebra of \mathfrak{g}_1 is the stabiliser of a flag, and as such is G_1 -conjugate to a block upper triangular subalgebra of \mathfrak{g}_1 . The nilpotent radical of a parabolic subalgebra in block form is the subalgebra obtained by replacing each diagonal block by a 0-block of the same size. Define the following subgroups of G_r :

$$U = U^0 = \mathfrak{A}^\times, \quad U^m = 1 + \mathfrak{P}^m, \text{ for } m \geq 1.$$

Let $e = e(\mathfrak{A})$ be the length of the flag in \mathfrak{g}_1 defining \mathfrak{A} . Then it can be shown that

$$(7.1) \quad \mathfrak{p}\mathfrak{A} = \mathfrak{A}\mathfrak{p} = \mathfrak{P}^e$$

and one can think of e as a ramification index. We have a filtration

$$U \supset U^1 \supset \dots \supset U^{er-1} \supset U^{er} = \{1\},$$

where the inclusions can be shown to be strict. It is also convenient to define $U^i = \{1\}$ for all $i > er$. Since \mathfrak{P} is a (two-sided) ideal in \mathfrak{A} , each group U^i is normal in U . Moreover, we have the commutator relation

$$[U^i, U^j] \subseteq U^{i+j}.$$

Thus in particular, the group U^i is abelian whenever $i \geq er/2$.

From now on, let $\beta \in \mathfrak{g}_r$ be a regular element and write $\bar{\beta}$ for its image in \mathfrak{g}_1 . We will associate a certain parahoric subalgebra to β (or rather, to the orbit of $\bar{\beta}$), which will be denoted by \mathfrak{A}_m . Let

$$\prod_{i=1}^h f_i(x)^{m_i} \in \mathbb{F}_q[x]$$

be the characteristic polynomial of $\bar{\beta}$, where the $f_i(x)$ are distinct and irreducible of degree d_i , for $i = 1, \dots, h$. This determines a partition of n :

$$\lambda = (d_1^{m_1}, \dots, d_h^{m_h}) = (\underbrace{d_1, d_1, \dots, d_1}_{m_1 \text{ times}}, \dots, \underbrace{d_h, d_h, \dots, d_h}_{m_h \text{ times}}).$$

We define $\mathfrak{A}_m \subseteq \mathfrak{g}_r$ to be the preimage of the standard parabolic subalgebra of \mathfrak{g}_1 corresponding to λ (i.e., the block upper-triangular subalgebra whose block sizes are given by λ , in the order given above). Moreover, we let $\mathfrak{A}_M = \mathfrak{g}_r = M_N(\mathfrak{o}_r)$ be the full matrix algebra. Let \mathfrak{P}_m and \mathfrak{P}_M be the corresponding ideals in \mathfrak{A}_m and \mathfrak{A}_M , respectively. For $* \in \{m, M\}$ we have the corresponding groups

$$U_* = U_*^0 = \mathfrak{A}_*^\times, \quad U_*^i = 1 + \mathfrak{P}_*^i, \quad \text{for } i \geq 1,$$

and the filtration

$$U_* \supset U_*^1 \supset \dots \supset U_*^{e_*r} = \{1\},$$

where $e_* = e(\mathfrak{A}_*)$. Note that $U_M^i = K^i$ and $e_M = 1$. The label m here stands for “minimal”, while M stands for “maximal”. From the definitions, we have

$$U_m/U_m^1 \cong \prod_{i=1}^h GL_{d_i}(\mathbb{F}_q)^{m_i},$$

$$U_M/U_M^1 \cong GL_N(\mathbb{F}_q).$$

Note that if $\bar{\beta}$ has irreducible characteristic polynomial, then $\mathfrak{A}_m = \mathfrak{g}_r$, and $U_m^i = K^i$ are the normal subgroups defined earlier.

By definition, we have $\mathfrak{A}_M \supseteq \mathfrak{A}_m$, and therefore $\mathfrak{P}_m \supseteq \mathfrak{P}_M$. The relations $\mathfrak{A}_M \supseteq \mathfrak{A}_m \supseteq \mathfrak{P}_m \supseteq \mathfrak{P}_M$ imply that for every $i \geq 1$, \mathfrak{P}_M^i is a two-sided ideal in \mathfrak{A}_m , so U_m normalises U_M^i . For $* \in \{m, M\}$, we can therefore define the following groups

$$C = C_{G_r}(\beta),$$

$$J_* = (C \cap U_*)U_*^{e_*l'},$$

$$J_*^1 = (C \cap U_*^1)U_*^{e_*l'},$$

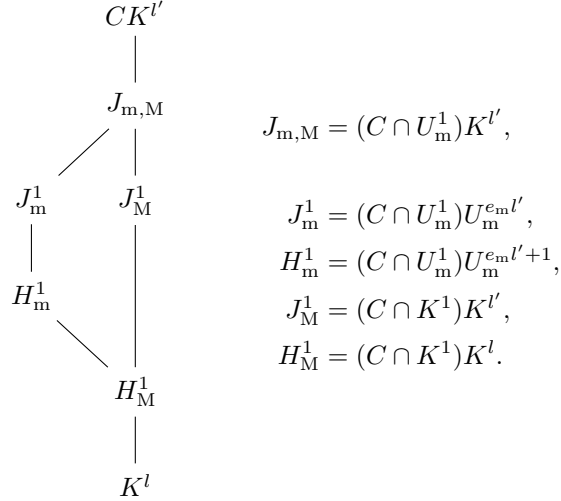
$$H_*^1 = (C \cap U_*^1)U_*^{e_*l'+1}.$$

Recall that since β is regular, C is abelian. Since $[U_*^1, U_*^{e_*l'}] \subseteq U_*^{e_*l'+1}$ and \mathfrak{A}_*^\times normalises $U_*^{e_*l'}$, the group J_* normalises both J_*^1 and H_*^1 . Moreover, we define the group

$$J_{m,M} = (C \cap U_m^1)K^{l'}.$$

We have the following diagram of subgroups, where the vertical and slanted lines denote inclusions (we have only indicated the inclusions which are relevant to us

and repeat the definitions of the groups, for the reader's convenience).



We explain the non-trivial inclusions in the above diagram. Since $\mathfrak{P}_m \supseteq \mathfrak{P}_M$, we have $U_m^1 \supseteq K^1$ and

$$U_m^{e_m l'+1} = 1 + \mathfrak{p}^{l'} \mathfrak{P}_m \supseteq 1 + \mathfrak{p}^{l'} \mathfrak{P}_M = K^l;$$

thus $H_m^1 \supseteq H_M^1$. Moreover,

$$U_m^{e_m l'} = 1 + \mathfrak{p}^{l'} \mathfrak{A}_m \subseteq 1 + \mathfrak{p}^{l'} \mathfrak{A}_M = K^{l'},$$

so $J_{m,M}$ contains both J_m^1 and J_M^1 as subgroups. We remark that J_M^1 is normal in $CK^{l'}$ since C normalises both K^1 and $K^{l'}$, and $[K^{l'}, K^1] \subseteq K^l \subseteq K^{l'}$.

The following lemma will be used in Step ?? of the construction we will outline below, and is the main reason why we work with the algebra \mathfrak{A}_m and its associated subgroups.

Lemma 7.1. *There exists a G_r -conjugate of β such that the group $J_{m,M}$ is a normal p -Sylow subgroup of $CK^{l'}$.*

We sketch the proof of this lemma. We first show that $J_{m,M}$ is normal in J_M . Since $C \cap \mathfrak{A}_M^\times$ normalises $J_{m,M}$ (C being abelian), it is enough to observe that $U_M^{e_M l'}$ normalises $J_{m,M}$ (in any finite group G with a normal subgroup N and a subgroup H , the group HN is normalised by N ; here G would be U_M). Write β_m for the image of β in U_m/U_m^1 . Then, up to conjugating β , we have

$$\beta_m = \underbrace{\beta_1 \oplus \cdots \oplus \beta_1}_{m_1 \text{ times}} \oplus \cdots \oplus \underbrace{\beta_h \oplus \cdots \oplus \beta_h}_{m_h \text{ times}},$$

where $\beta_i \in M_{d_i}(\mathbb{F}_q)$, and d_i and m_i are as in the partition λ above. With β_m of the above form, one can show that β being regular implies that $C \subseteq U_m$, so we have an isomorphism

$$CK^{l'}/J_{m,M} \cong \frac{C}{(C \cap U_m^1)(C \cap K^{l'})} = \frac{C \cap U_m}{(C \cap U_m^1)}.$$

Then the isomorphism $U_m/U_m^1 \cong \prod_{i=1}^h GL_{d_i}(\mathbb{F}_q)^{m_i}$ induces an isomorphism

$$\frac{C \cap U_m}{C \cap U_m^1} \cong \prod_{i=1}^h C_{GL_{d_i}(\mathbb{F}_q)}(\beta_i)^{m_i}.$$

Each β_i has irreducible characteristic polynomial over \mathbb{F}_q , so $\mathbb{F}_q[\beta_i]/\mathbb{F}_q$ is an extension of degree d_i . Since $C_{GL_{d_i}(\mathbb{F}_q)}(\beta_i) = \mathbb{F}_q[\beta_i]^\times$, we conclude that p does not divide the order of $C_{GL_{d_i}(\mathbb{F}_q)}(\beta_i)$. Therefore, p does not divide the order of $\frac{C}{C \cap U_m^1}$, so $J_{m,M}$ is a p -Sylow subgroup of $CK^{l'}$ (in fact the unique p -Sylow subgroup, since it is normal).

7.2. Characters. Let $\psi : F \rightarrow \mathbb{C}^\times$ be as in Section 3. Let $\mathfrak{A}, \mathfrak{P}$, and $U^m, m \geq 0$ be the objects associated to an arbitrary flag of length e , as in Section 7.1. Let n and m be two integers such that $e(r-1) + 1 \geq n > m \geq n/2 > 0$. Then U^m/U^n is abelian, and we have an isomorphism

$$\mathfrak{P}^m/\mathfrak{P}^n \xrightarrow{\sim} U^m/U^n, \quad x + \mathfrak{P}^n \mapsto (1+x)U^n.$$

Each $a \in \mathfrak{g}_r$ defines a character $\mathfrak{g}_r \rightarrow \mathbb{C}^\times$ via $x \mapsto \psi(\text{tr}(ax))$, and this defines an isomorphism $\mathfrak{g}_r \rightarrow \text{Irr}(\mathfrak{g}_r)$. For any subgroup S of \mathfrak{g}_r , define

$$S^\perp = \{x \in \mathfrak{g}_r \mid \psi(\text{tr}(xS)) = 1\}.$$

Using the isomorphism $\mathfrak{g}_r \rightarrow \text{Irr}(\mathfrak{g}_r)$, we can identify S^\perp with the group of characters of \mathfrak{g}_r which are trivial on S .

For any $\beta \in \mathfrak{P}^{e(r-1)+1-n}$ define a character $\psi_\beta : U^m \rightarrow \mathbb{C}^\times$ by

$$\psi_\beta(1+x) = \psi(\varpi^{-r} \text{tr}(\beta x)).$$

Lemma 7.2. *Let $e(r-1) + 1 \geq n > m \geq n/2 > 0$. Then*

(i) *For any integer i such that $0 \leq i \leq e(r-1) + 1$, we have*

$$(\mathfrak{P}^i)^\perp = \mathfrak{P}^{e(r-1)+1-i}.$$

(ii) *The map $\beta \mapsto \psi_\beta$ induces an isomorphism*

$$\mathfrak{P}^{e(r-1)+1-n}/\mathfrak{P}^{e(r-1)+1-m} \xrightarrow{\sim} \text{Irr}(U^m/U^n).$$

We omit the proof of this lemma, and only remark that the first part essentially follows from the observation that $j = e(r-1) + 1$ is the smallest integer such that \mathfrak{P}^j is strictly block-upper triangular mod \mathfrak{p}^r . Indeed, $\mathfrak{P}^{e(r-1)+1} = \mathfrak{p}^{r-1}\mathfrak{P}$, and \mathfrak{P} is strictly block-upper mod \mathfrak{p} . This implies that $\mathfrak{P}^\perp = \mathfrak{P}^{e(r-1)+1}$, and the general case follows similarly.

As a special case of the above, suppose that $e = 1$, so that $\mathfrak{A} = \mathfrak{g}_r$ and $U^m = K^m = 1 + \mathfrak{p}^m \mathfrak{g}_r$. For any $r = n > m \geq r/2$ and $\beta \in \mathfrak{g}_r$, we have a character $\psi_\beta : K^m \rightarrow \mathbb{C}^\times$ defined as above, and the isomorphism of Lemma 7.2 (ii) becomes

$$\mathfrak{g}_r/\mathfrak{p}^{r-m} \mathfrak{g}_r \xrightarrow{\sim} \text{Irr}(K^m),$$

which agrees with the considerations in Section 3.

7.3. Construction of representations. For our fixed arbitrary regular element $\beta \in \mathfrak{g}_r$, we start with the character ψ_β of K^l , and construct all the irreducible representations of CK^l which contain ψ_β . Theorem 3.2(ii) then yields all the irreducible representations of G_r with β in their orbits. The construction consists of a number of steps. For each step we indicate some of the details involved.

Some of the steps can be carried out for the groups arising from the algebras \mathfrak{A}_m and \mathfrak{A}_M simultaneously. For this purpose, we will let \mathfrak{A} denote either \mathfrak{A}_m or \mathfrak{A}_M , and let \mathfrak{P} be the radical in \mathfrak{A} , with “ramification index” e . The associated subgroups will be denoted by U^i , H^1 , J^1 .

Step 1: Show that ψ_β has an extension θ_M to H_M^1 . Show that θ_M has an extension θ_m to H_m^1 .

By Lemma 7.2(ii), if we take

$$m = el' + 1, \quad n = 2m - 1 = e(r - 1) + 1,$$

then β , or rather the coset $\beta + \mathfrak{P}^{el'}$, defines a character on U^m , trivial on U^n by the same formula as the one defining ψ_β . Since $\mathfrak{P}^{el'} = \mathfrak{p}^{l'}\mathfrak{A}$, we have a map

$$\mathfrak{A}/\mathfrak{P}^{el'} \longrightarrow \mathfrak{g}_r/\mathfrak{p}^{l'}\mathfrak{g}_r,$$

which sends the coset $\beta + \mathfrak{P}^{el'}$ to $\beta + \mathfrak{p}^{l'}\mathfrak{g}_r$. Thus the different choices of lift of the latter coset give the different choices of extension of ψ_β to $U^{el'+1}$. Our element $\beta \in \mathfrak{A}$ therefore gives rise to an extension (which we still denote by ψ_β) of ψ_β to $U^{el'+1}$, defined by

$$\psi_\beta(1 + x) = \psi(\varpi^{-r} \operatorname{tr}(\beta x)), \quad \text{for } x \in \mathfrak{P}^{el'+1}.$$

We now show the existence of the extensions θ_M and θ_m . If $c \in C \cap U^1$ and $x \in \mathfrak{P}^{el'+1}$, then

$$\begin{aligned} [c, 1 + x] &\in c(1 + x)c^{-1}(1 - x + \mathfrak{P}^{e(r-1)+2}) \\ &= 1 + cxc^{-1} - x + \mathfrak{P}^{e(r-1)+2}. \end{aligned}$$

By Lemma 7.2(i), since $\beta \in \mathfrak{A}$, we have

$$(7.2) \quad U^{e(r-1)+1} \subseteq \operatorname{Ker} \psi_\beta,$$

so

$$\psi_\beta([c, 1 + x]) = \psi(\varpi^{-r} \operatorname{tr}(\beta(cxc^{-1} - x))) = \psi(\varpi^{-r} \operatorname{tr}(c\beta xc^{-1} - \beta x)) = 1,$$

where we have used that c commutes with β .

Thus $C \cap U^1$ stabilises the character ψ_β on $U^{el'+1}$, and since $C \cap U^1$ is abelian, this implies that ψ_β extends to $H^1 = (C \cap U^1)U^{el'+1}$. We fix an extension θ_M to H_M^1 and an extension of θ_M to H_m^1 , denoted θ_m .

Step 2: For $* \in \{m, M\}$, construct the irreducible representations η_* of J_*^1 containing θ_* . In particular, show that there exists a unique representation η_M of J_M^1 containing θ_M .

As in the previous step, we will treat both cases simultaneously, denoting either θ_m or θ_M by θ . We outline the ingredients needed for this. First note that θ is

stabilised by J^1 : Indeed, it is enough to show that $U^{el'}$ stabilises θ . For $x \in \mathfrak{P}^{el'}$, $c \in (C \cap U^1)$ and $y \in \mathfrak{P}^{el'+1}$, we have

$$\begin{aligned} [1+x, c(1+y)] &\in (1+x)c(1+y)(1-x+x^2+\mathfrak{P}^{e(r-1)+1})(1-y+\mathfrak{P}^{e(r-1)+1})c^{-1} \\ &\subseteq (c+xc-cx+cy)(1-y)c^{-1}+\mathfrak{P}^{e(r-1)+1} \\ &\subseteq 1+x-cxc^{-1}+\mathfrak{P}^{e(r-1)+1}. \end{aligned}$$

Hence, since ψ_β is trivial on $U^{e(r-1)+1}$ (see (7.2)) and c commutes with β , we have

$$\theta([1+x, c(1+y)]) = \psi_\beta([1+x, c(1+y)]) = \psi(\varpi^{-r} \operatorname{tr}(c\beta xc^{-1} - \beta x)) = 1.$$

Next, we have

$$J^1/H^1 \cong \frac{U^{el'}}{(C \cap U^{el'})U^{el'+1}},$$

and $U^{el'}/U^{el'+1}$ is isomorphic to a subgroup of $\mathfrak{g}_1 = M_N(\mathbb{F}_q)$. Thus J^1/H^1 is a quotient of an elementary abelian p -group and has the structure of a finite dimensional \mathbb{F}_q -vector space. Define the alternating bilinear form

$$h_\beta : J^1/H^1 \times J^1/H^1 \longrightarrow \mathbb{C}^\times, \quad h_\beta(xH^1, yH^1) = \theta([x, y]) = \psi_\beta([x, y]).$$

Note that $[J^1, J^1] \subseteq U^{el'+1}$, so we have $\theta([x, y]) = \psi_\beta([x, y])$.

Let \overline{R}_β be the radical of the form h_β , and let \overline{W}_β be a maximal isotropic subspace (if we need to specify which parabolic subalgebra \mathfrak{A}_* we are working with, we will write $\overline{R}_{\beta,*}$ and $\overline{W}_{\beta,*}$, for $*$ in $\{\mathfrak{m}, \mathfrak{M}\}$). Let R_β and W_β denote the preimages of \overline{R}_β and \overline{W}_β under the map $J^1 \rightarrow J^1/H^1$, respectively. For our purposes, we need to determine the order of the group W_β and this can be done by determining the order of R_β , or equivalently, the dimension of \overline{R}_β (as a vector space over \mathbb{F}_q). Consider the map

$$\rho : U^{el'} \longrightarrow U^{el'}/U^{el'+1} \xrightarrow{\sim} \mathfrak{A}/\mathfrak{P},$$

where the isomorphism is given by $(1 + \varpi^{el'} x)U^{el'+1} \mapsto x + \mathfrak{P}$. Let $\bar{\beta}$ denote the image of β in $\mathfrak{A}/\mathfrak{P}$ under this map. One can then show that

$$R_\beta = (C \cap U^1) \cdot \rho^{-1}(C_{\mathfrak{A}/\mathfrak{P}}(\bar{\beta})).$$

A general result then says that there exists an extension θ' of θ to R_β , and, for each such extension θ' , a unique $\eta \in \operatorname{Irr}(J^1 \mid \theta')$. Indeed, one shows that there exists an extension θ'' of θ' to W_β , that $\eta := \operatorname{Ind}_{W_\beta}^{J^1} \theta''$ is irreducible and that η is independent of the choice of extension θ'' to W_β (cf. Section 6.1). In particular, it turns out that $R_{\beta, \mathfrak{M}} \subseteq H_{\mathfrak{M}}^1$, so there is no choice for θ' in this case, and hence there exists a unique $\eta_{\mathfrak{M}} \in \operatorname{Irr}(J_{\mathfrak{M}}^1 \mid \theta_{\mathfrak{M}})$.

Step 3: Show that there exists an extension $\hat{\eta}_{\mathfrak{M}}$ of $\eta_{\mathfrak{M}}$ to $J_{\mathfrak{M}}$.

This step can be seen as the reason for involving the “auxiliary” path through $H_{\mathfrak{m}}^1$ and $J_{\mathfrak{m}}^1$. In the previous step, we constructed an irreducible representation $\eta_{\mathfrak{m}}$ of $J_{\mathfrak{m}}^1$ containing $\theta_{\mathfrak{m}}$. We now need to determine the dimension of the induced representation

$$\eta := \operatorname{Ind}_{J_{\mathfrak{m}}^1}^{J_{\mathfrak{m}, \mathfrak{M}}} \eta_{\mathfrak{m}}.$$

The order of $R_{\beta, \mathfrak{m}}$, can be used to calculate the dimension of $\eta_{\mathfrak{m}}$, indeed $\dim \eta_{\mathfrak{m}} = [J_{\mathfrak{m}}^1 : R_{\beta, \mathfrak{m}}]^{1/2}$, so

$$\dim \eta = [J_{\mathfrak{m}}^1 : R_{\beta, \mathfrak{m}}]^{1/2} [J_{\mathfrak{m}, \mathfrak{M}} : J_{\mathfrak{m}}^1].$$

Comparing this with the dimension of η_M , which is $[J_M^1 : H_M^1]^{1/2} = q^{N(N-1)/2}$, it turns out that $\dim \eta_M = \dim \eta$. Then, since the restricted representation $\eta|_{J_M^1}$ contains θ_M on further restriction to H_M^1 , and η_M is the unique representation of J_M^1 with this property, it follows that η contains η_M on restriction to J_M^1 . The equality of the dimensions then forces $\eta|_{J_M^1} = \eta_M$ (and in particular, η is irreducible).

Furthermore, one can show that all of $CK^{l'}$ stabilises the character θ_M . Since $J_{m,M}$ is a p -Sylow subgroup in $CK^{l'}$ by Lemma 7.1 and η_M extends to $J_{m,M}$, it then follows from [14, Corollary 11.31] and a theorem of Gallagher [7, Theorem 6] that η_M has an extension $\hat{\eta}_M$ to $CK^{l'}$ (the same extension result was used in the end of Section 6 for the extension from J_β to C_{G_r}).

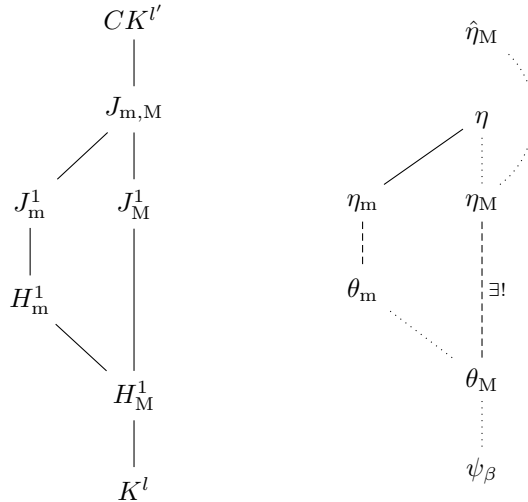
Note that η_m is not the only representation containing θ_m , and therefore η is not unique. This does not matter for us, since we are only interested in proving that η_M has an extension, so we only need one representation η .

We also remark that even though both η and $\hat{\eta}_M$ are extensions of η_M , we do not know (and do not need to know) whether $\hat{\eta}_M$ is an extension of η .

Step 4: The final step in the construction is to note that every irreducible representation of $CK^{l'}$ which contains ψ_β is of the form $\hat{\eta}_M$ for some choice of extension θ_M of ψ_β and some choice of extension $\hat{\eta}_M$ of η_M , and that distinct choices of θ_M , as well as distinct choices of extensions $\hat{\eta}_M$ of η_M , give rise to distinct representations of $CK^{l'}$.

By a standard result in Clifford theory (Lemma 3.2) we have a one to one correspondence between $\text{Irr}(CK^{l'} \mid \psi_\beta)$ and $\text{Irr}(G_r \mid \psi_\beta)$ given by induction. Thus, we have constructed all the irreducible representations of G_r with β in their orbits.

Schematically, the construction is illustrated by the following diagrams (dotted lines are extensions, dashed are Heisenberg lifts, and solid one between η_m and η is an induction):



8. OPEN PROBLEMS

We close with a non-exhaustive list of open problems in the representation theory of $G_r = GL_N(\mathfrak{o}_r)$. Several other problems are suggested in [2, Section 1.6].

8.1. Beyond GL_N . It is natural to ask whether it is possible to construct regular representations of reductive groups over \mathfrak{o}_r other than GL_N . As we have already mentioned, [18] constructs regular representations for $\mathrm{SL}_N(\mathfrak{o}_r)$, $p \nmid N$, as well as for unitary groups. These cases are relatively close to GL_N , but one may expect that it is possible to construct the regular representations of $G(\mathfrak{o}_r)$ whenever G is a sufficiently nice reductive group scheme over \mathfrak{o} , for example when the derived group of G is simply connected and p is a very good prime. The first step is to show that under some hypotheses on G , any $\beta \in \mathrm{Lie}(G)(\mathfrak{o}_r)$ such that $\bar{\beta} \in \mathrm{Lie}(G)(\mathbb{F}_q)$ is regular, will have abelian centraliser in $G(\mathfrak{o}_r)$ and the surjective mapping property of centralisers under reduction maps.

8.2. Beyond regular representations. Hill's construction of strongly semisimple representations (see Section 5) shows that Clifford theoretic methods can be used to construct some non-regular representations of $\mathrm{GL}_N(\mathfrak{o}_r)$, up to knowledge of all the irreducible representations of $\mathrm{GL}_{N'}(\mathfrak{o}_{r'})$ for $N' < N$, $r' < r$. Is there a uniform construction which includes the regular representations and the strongly semisimple representations (and perhaps others)?

8.3. Relation with supercuspidal types. Henniart [3] and Paskunas [26] have shown that every supercuspidal representation of $\mathrm{GL}_N(F)$ has a unique *type* on $\mathrm{GL}_N(\mathfrak{o})$. It would be interesting to identify the regular representations which are supercuspidal types and determine what they map to under the inertial Langlands correspondence.

8.4. Onn's conjectures. For each integer $n \geq 1$, let

$$r_n = r_n(G_r) = \#\{\pi \in \mathrm{Irr}(G_r) \mid \dim \pi = n\}.$$

The experience with the known cases of $\mathrm{GL}_2(\mathfrak{o}_r)$ [30, 25], $\mathrm{GL}_3(\mathfrak{o}_r)$ [2] and the regular representations of $\mathrm{GL}_N(\mathfrak{o}_r)$, suggests that r_n , as a function of \mathfrak{o}_r , is rather well behaved. More precisely, in all known cases, it is a polynomial over \mathbb{Q} in the size q of the residue field, independent of the compact DVR \mathfrak{o} , as long as the residue field is \mathbb{F}_q . Moreover, the dimensions of the known representations of $\mathrm{GL}_N(\mathfrak{o}_r)$ are given by polynomials in q , and one may ask whether this is true in general. In [25] Onn made the following conjectures, which we paraphrase slightly and state only for $\mathrm{GL}_N(\mathfrak{o}_r)$:

Conjecture (Onn).

- (i) Suppose \mathfrak{o} and \mathfrak{o}' are two compact DVRs with maximal ideals \mathfrak{p} and \mathfrak{p}' , respectively, such that $|\mathfrak{o}/\mathfrak{p}| = |\mathfrak{o}'/\mathfrak{p}'|$. Then there is an isomorphism of group algebras

$$\mathbb{C}[\mathrm{GL}_N(\mathfrak{o}_r)] \cong \mathbb{C}[\mathrm{GL}_N(\mathfrak{o}'_r)].$$

- (ii) For any $n \geq 1$ there exists a polynomial $p_n(x) \in \mathbb{Q}[x]$ such that for any compact DVR \mathfrak{o} we have

$$r_n(\mathrm{GL}_N(\mathfrak{o}_r)) = p_n(q),$$

where $q = |\mathfrak{o}/\mathfrak{p}|$.

- (iii) There exist finitely many polynomials $d_1(x), \dots, d_h(x) \in \mathbb{Z}[x]$ with $\deg d_i \leq \binom{N}{2}r$, such that for any compact DVR \mathfrak{o} we have

$$\{\dim \pi \mid \pi \in \mathrm{Irr}(\mathrm{GL}_N(\mathfrak{o}_r)), \pi \text{ primitive}\} = \{d_1(q), \dots, d_h(q)\},$$

where $q = |\mathfrak{o}/\mathfrak{p}|$.

Note that part (ii) of this conjecture implies part (i).

REFERENCES

- [1] N. Avni, B. Klopsch, U. Onn, and C. Voll, *Representation zeta functions of compact p -adic analytic groups and arithmetic groups*, Duke Math. J. **162** (2013), no. 1, 111–197.
- [2] N. Avni, B. Klopsch, U. Onn, and C. Voll, *Similarity classes of integral p -adic matrices and representation zeta functions of groups of type A_2* , Proc. Lond. Math. Soc. (3) **112** (2016), no. 2, 267–350.
- [3] C. Breuil and A. Mézard, *Multiplicités modulaires et représentations de $GL_2(\mathbf{Z}_p)$ et de $Gal(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ en $l = p$* , Duke Math. J. **115** (2002), no. 2, 205–310, with an appendix by G. Henniart.
- [4] C. J. Bushnell and P. C. Kutzko, *The Admissible Dual of $GL(N)$ via Compact Open Subgroups*, volume 129 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, 1993.
- [5] Z. Chen and A. Stasinski, *The algebraisation of higher Deligne–Lusztig representations*, arXiv:1604.01615.
- [6] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161.
- [7] P. X. Gallagher, *Group characters and normal Hall subgroups*, Nagoya Math. J. **21** (1962), 223–230.
- [8] P. Gérardin, *Construction de séries discrètes p -adiques*, Lecture Notes in Mathematics 462, Springer-Verlag, Berlin, 1975.
- [9] J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. **80** (1955), 402–447.
- [10] G. Hill, *A Jordan decomposition of representations for $GL_n(\mathcal{O})$* , Comm. Algebra **21** (1993), no. 10, 3529–3543.
- [11] G. Hill, *On the nilpotent representations of $GL_n(\mathcal{O})$* , Manuscripta Math. **82** (1994), no. 3-4, 293–311.
- [12] G. Hill, *Regular elements and regular characters of $GL_n(\mathcal{O})$* , J. Algebra **174** (1995), no. 2, 610–635.
- [13] G. Hill, *Semisimple and cuspidal characters of $GL_n(\mathcal{O})$* , Comm. Algebra **23** (1995), no. 1, 7–25.
- [14] I. M. Isaacs, *Character Theory of Finite Groups*, Pure and Applied Mathematics, No. 69, Academic Press, New York, 1976.
- [15] A. Jaikin-Zapirain, *Zeta function of representations of compact p -adic analytic groups*, J. Amer. Math. Soc. **19** (2006), no. 1, 91–118.
- [16] H. D. Kloosterman, *The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups. I*, Ann. of Math. (2) **47** (1946), 317–375.
- [17] H. D. Kloosterman, *The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups. II*, Ann. of Math. (2) **47** (1946), 376–447.
- [18] R. Krakovski, U. Onn, and P. Singla, *Regular characters of groups of type A_n over discrete valuation rings* arXiv:1604.00712.
- [19] G. Lusztig, *Representations of reductive groups over finite rings*, Represent. Theory **8** (2004), 1–14.
- [20] S. V. Nagornyj, *Complex representations of the group $GL(2, \mathbf{Z}/p^n\mathbf{Z})$* , Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **64** (1976), 95–103, 161, rings and modules.
- [21] S. V. Nagornyj, *Complex representations of the general linear group of degree three modulo a power of a prime*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **75** (1978), 143–150, 197–198, rings and linear groups.
- [22] A. Nobs, *Die irreduziblen Darstellungen der Gruppen $SL_2(\mathbf{Z}_p)$, insbesondere $SL_2(\mathbf{Z}_2)$. I*, Comment. Math. Helv. **51** (1976), no. 4, 465–489.
- [23] A. Nobs, *Die irreduziblen Darstellungen von $GL_2(\mathbf{Z}_p)$, insbesondere $GL_2(\mathbf{Z}_2)$* , Math. Ann. **229** (1977), no. 2, 113–133.
- [24] A. Nobs and J. Wolfart, *Die irreduziblen Darstellungen der Gruppen $SL_2(\mathbf{Z}_p)$, insbesondere $SL_2(\mathbf{Z}_p)$. II*, Comment. Math. Helv. **51** (1976), no. 4, 491–526.
- [25] U. Onn, *Representations of automorphism groups of finite \mathfrak{o} -modules of rank two*, Adv. in Math. **219** (2008), no. 6, 2058–2085.

- [26] V. Paskunas, *Unicity of types for supercuspidal representations of GL_N* , Proc. London Math. Soc. (3) **91** (2005), no. 3, 623–654.
- [27] J. A. Shalika, *Representation of the two by two unimodular group over local fields*, in *Contributions to automorphic forms, geometry, and number theory*, pp. 1–38, Johns Hopkins Univ. Press, Baltimore, MD, 2004.
- [28] T. Shintani, *On certain square-integrable irreducible unitary representations of some p -adic linear groups*, J. Math. Soc. Japan **20** (1968), 522–565.
- [29] A. J. Silberger, *PGL_2 over the p -adics: its representations, spherical functions, and Fourier analysis*, Lecture Notes in Mathematics, Vol. 166, Springer-Verlag, Berlin, 1970.
- [30] A. Stasinski, *The smooth representations of $GL_2(\mathcal{O})$* , Comm. Algebra **37** (2009), 4416–4430.
- [31] A. Stasinski, *Unramified representations of reductive groups over finite rings*, Represent. Theory **13** (2009), 636–656.
- [32] A. Stasinski and S. Stevens, *The regular representations of GL_N over finite local principal ideal rings*, arXiv:1611.04796.
- [33] K. Takase, *Regular characters of $GL_n(\mathcal{O})$ and Weil representations over finite fields*, J. Algebra **449** (2016), 184–213.
- [34] S. Tanaka, *On irreducible unitary representations of some special linear groups of the second order. I, II*, Osaka J. Math. **3** (1966), 217–227; 229–242.
- [35] S. Tanaka, *Irreducible representations of the binary modular congruence groups mod p^λ* , J. Math. Kyoto Univ. **7** (1967), 123–132.

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